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# Shifted $1/N$ Expansion Within the Framework of the $SO(2,1)$ Pseudo-spin Formalism

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**Abstract** The shifted $1/N$ expansion is organized within the framework of the $SO(2,1)$ pseudo-spin formalism and the method is formulated to derive the ground state energies of a class of power law potentials.

**Keywords** Shifted large-$N$ expansion · Pseudo-spin formalism · Power-law potentials

The method of large-$N$ expansion has won acceptance as a powerful non-perturbative approach in quantum mechanics, atomic physics, statistical mechanics and field theory (for a review see [1–7]). Notwithstanding its proven efficacy in dealing with a wide variety of problems [8–32] the method is at times plagued with slow convergence. To improve the situation, Sukhatme and Imbo [33] have developed the so called shifted $1/N$ expansion scheme introducing an extra degree of freedom, called the shift parameter. Sukhatme and collaborators [34] have given a prescription for the choice of the shift parameter which is physically motivated and developed their theory in the framework of the perturbed oscillator technique. The shifted $1/N$ expansion method [33] has been used by several authors with overwhelming success [35–49]. There have also been several other formulations [50–54] to obtain improved convergence in the $1/N$ expansion.

In the present paper we have made an attempt to introduce the idea of the shifted $1/N$ expansion of Sukhatme and Imbo [34] in a wider framework namely in the pseudo-spin formulation [55, 56]. It is well-known that the pseudo-spin formalism is a very powerful and versatile technique which finds applications in various branches of physics [57–60] and therefore it is expected that the shifted $1/N$ expansion in the pseudo-spin formalism would have a much wider applicability. We shall formulate the theory for a class of spherically symmetric potentials and compare the results with those obtained by Sukhatme and Imbo [33] with the perturbed oscillator method.

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We consider an $N$-dimensional hamiltonian with a spherically symmetric potential of the form $\frac{1}{2}{\omega_0}^2 r^2 + V(r^2)$. The hamiltonian reads (in units $\hbar = m = 1$)

$$H = \frac{1}{2}p^2 + \frac{1}{2}{\omega_0}^2 r^2 + V(r^2), \tag{1}$$

where $p^2 = \sum_i p_i p_i, i = 1, 2, 3, \ldots, N$, and $r^2 = \sum_i x_i x_i$, where $\vec{r} = (x_1, x_2, x_3, \ldots, x_N)$. Introducing the creation and annihilation operators, $a^+$ and $a$ in the usual manner:

$$a_i = \frac{\hat{p}_i}{\sqrt{2\omega}} - i\sqrt{\frac{\omega}{2}}x_i; \qquad {a_i}^\dagger = \frac{\hat{p}_i}{\sqrt{2\omega}} + i\sqrt{\frac{\omega}{2}}x_i, \tag{2}$$

where $\omega$ is a free parameter to be chosen later, one can construct three $O(N)$ invariant bilinear operators

$$\hat{K} = \frac{1}{2}\sum_{i=1}^{N} a_i a_i, \qquad \hat{K}^\dagger = \frac{1}{2}\sum_{i=1}^{N} {a_i}^\dagger {a_i}^\dagger, \qquad \hat{K}_0 = \frac{1}{4}\sum_{i=1}^{N}[{a_i}^\dagger a_i + a_i {a_i}^\dagger], \tag{3}$$

which close the $SO(2, 1)$ algebra:

$$[\hat{K}, \hat{K}^\dagger] = 2\hat{K}_0, \qquad [\hat{K}_0, \hat{K}] = -\hat{K}, \qquad [\hat{K}_0, \hat{K}^\dagger] = \hat{K}^\dagger. \tag{4}$$

The Casimir invariant of this algebra is given by

$$\hat{C} = \hat{K}_0^2 - \frac{1}{2}(\hat{K}\hat{K}^\dagger + \hat{K}^\dagger\hat{K}) = \frac{L^2}{4} + \frac{N}{4}\left(\frac{N}{4} - 1\right), \tag{5}$$

where $L^2$ is the $N$-dimensional angular momentum operator having the eigen value $l(l + N - 2)$, $l = 0, 1, 2, \ldots$. The eigen value of $\hat{C}$ can be written as $C = k'(k' - 1)$, where $k' = (N + 2l)/4$. In terms of the pseudo-spin generators $\hat{K}$, $\hat{K}^\dagger$ and $\hat{K}_0$, the hamiltonian (1) reads

$$H = \omega\left(1 + \frac{{\omega_0}^2}{\omega^2}\right)\hat{K}_0 + \frac{\omega}{2}\left(1 - \frac{{\omega_0}^2}{\omega^2}\right)(\hat{K} + \hat{K}^\dagger) + V\left(\frac{1}{\omega}[2\hat{K}_0 - \hat{K} + \hat{K}^\dagger]\right). \tag{6}$$

We shall consider here spherically symmetric potential problems. To eliminate the angular coordinates we perform following MP generalized Holstein-Primakoff transformations:

$$\hat{K}_0 = k' + \zeta^\dagger\zeta, \qquad \hat{K} = (2k' + \zeta^\dagger\zeta)^{1/2}\zeta, \qquad \hat{K}^\dagger = \zeta^\dagger(2k' + \zeta^\dagger\zeta)^{1/2}, \tag{7}$$

where $[\zeta, \zeta^\dagger] = 1$. To concretize the problem let us consider a specific case, namely, $V(r^2) = 2\nu^2 r^{2\nu}$, where $\nu$ may be positive or negative but the total potential is always attractive. The hamiltonian then reads

$$\begin{aligned} H_{k'} = {} & \omega\left(1 + \frac{{\omega_0}^2}{\omega^2}\right)(k' + \zeta^\dagger\zeta) + \frac{\omega}{2}\left(1 - \frac{{\omega_0}^2}{\omega^2}\right)[(2k' + \zeta^\dagger\zeta)^{1/2}\zeta + \zeta^\dagger(2k' + \zeta^\dagger\zeta)^{1/2}] \\ & + \frac{2\nu e^2}{\omega^\nu}[2k' + 2\zeta^\dagger\zeta - (2k' + \zeta^\dagger\zeta)^{1/2}\zeta - \zeta^\dagger(2k' + \zeta^\dagger\zeta)^{1/2}]^\nu, \end{aligned} \tag{8}$$

where $k' = (N + 2l)/4, l = 0, 1, 2, \ldots$. We now introduce the expansion parameter $\alpha = (2k' - s)^{-1/2}$, where $s$ is called the shift parameter. Next performing the canonical transformation

$$\zeta = \left[\xi + \frac{\sinh\phi}{\alpha}\right], \qquad [\zeta, \zeta^\dagger] = 1, \tag{9}$$

with $\phi$ as a free parameter, (6) is expanded in powers of $\alpha$. To obtain the conditions to fix the parameters $\omega$ and $\phi$, we equate the coefficients of $(\xi^\dagger + \xi)$ and $({\xi^\dagger}^2 + \xi^2)$ to order $\alpha^0$ be zero. This gives

$$\omega^2 e^{4\phi} - {\omega_0}^2 = 4\nu^2 e^2 \left[\frac{\omega e^{2\phi}}{2k' - s}\right]^{1-\nu} \tag{10}$$

$$2\cosh^2\phi = [4 + \omega^2(\nu - 1)(e^{4\phi} - 1)]^{1/2} \tag{11}$$

The hamiltonian then takes the form

$$H_{k'} = h^{(0)} + \alpha h^{(1)} + \alpha^2 h^{(2)} + 0(\alpha^3), \tag{12}$$

where

$$h^{(0)} = \alpha^{-2} T_1 + \alpha^0 (T_2 \xi^\dagger \xi + T_3) \tag{13}$$

with

$$T_1 = \frac{\omega e^{2\phi}(1 + \nu - \cosh^4\phi)}{\nu}, \qquad T_2 = 2\omega e^{2\phi}\cosh^2\phi,$$
$$T_3 = \omega e^{2\phi}(\sinh^2\phi + s) \tag{14}$$

$$h^{(1)} = ({\xi^\dagger}^3 + \xi^3)A_1 + (\xi^\dagger \xi^2 + {\xi^\dagger}^2 \xi)A_2 + (\xi^\dagger + \xi)A_3 \tag{15}$$

with

$$A_1 = \frac{\tanh^3\phi}{4\cosh\phi} + \frac{(\nu - 1)(\omega^2 e^{4\phi} - {\omega_0}^2)}{4\omega e^{2\phi}\cosh\phi}\left[\frac{1}{2}\tanh\phi(2 - \tanh^2\phi)e^{2\phi} - \frac{(\nu - 2)}{3\cosh^2\phi}\right] \tag{16}$$

$$A_2 = \frac{\tanh\phi(3\tanh^\phi - 4)}{4\cosh\phi} + \frac{(\nu - 1)(\omega^2 e^{4\phi} - \omega^2)}{4\omega^2 e^{4\phi}\cosh\phi}\left[\left(5\tanh\phi - \frac{3}{2}\tanh^2\phi - 4\right)e^{2\phi} - \frac{(\nu - 2)}{\cosh^2\phi}\right] \tag{17}$$

$$A_3 = 3A_1 - \frac{(\nu - 1)(\omega^2 e^{4\phi} - {\omega_0}^2)e^{2\phi}}{2\omega^2 e^{4\phi}\cosh\phi}(1 + s - s\tanh\phi) - \frac{s\tanh\phi}{\cosh\phi} \tag{18}$$

and

$$h^{(2)} = -[\sinh(2\phi)\cosh\phi]\,G + \frac{(\omega^2 e^{4\phi} - {\omega_0}^2)}{2\omega}\left[e^{2\phi}\frac{(\nu - 1)}{2}\left(\frac{\tanh^3\phi}{4} - s\tanh\phi + s\right)^2 + \frac{(\nu - 1)e^{2\phi}}{2}\left(2 - 2\tanh\phi + \frac{1}{2}\tanh^3\phi\right)^2 (\xi^\dagger\xi)^2\right.$$

$$
\begin{aligned}
&+\frac{(\nu-1)e^{2\phi}}{2}\left(\frac{\tanh\phi}{2}-\frac{\tanh^3\phi}{4}\right)^2(\xi^{\dagger 2}+\xi^2)^2\\
&+\frac{(\nu-1)}{2}[(\xi^\dagger+\xi)F+F(\xi^\dagger\\
&+\xi)]+(\nu-1)e^{2\phi}\left(\frac{\tanh^3\phi}{4}-s\,\tanh\phi+s\right)\left(2-2\tanh\phi+\frac{\tanh^3\phi}{2}\right)\xi^\dagger\xi\\
&-(\nu-1)e^{2\phi}\left(\frac{\tanh^3\phi}{4}-s\,\tanh\phi+s\right)\left(\frac{\tanh\phi}{2}-\frac{\tanh^3\phi}{4}\right)(\xi^{\dagger 2}+\xi^2)\\
&-\frac{(\nu-1)(\nu-2)e^{2\phi}}{2\cosh^2\phi}\left(\frac{\tanh^3\phi}{4}-s\,\tanh\phi+s\right)(\xi^\dagger+\xi)^2\\
&+\frac{(\nu-1)(\nu-2)}{6\cosh^2\phi}\left(2-2\,\tanh\phi+\frac{1}{2}\tanh^3\phi\right)[(\xi^\dagger+\xi)^2\xi^\dagger\xi\\
&+(\xi^\dagger+\xi)\xi^\dagger\xi(\xi^\dagger+\xi)+\xi^\dagger\xi(\xi^\dagger+\xi)^2]\\
&+\frac{(\nu-1)(\nu-2)}{6\cosh^2\phi}\left(\frac{\tanh\phi}{2}-\frac{\tanh^3\phi}{4}\right)[(\xi^\dagger+\xi)^2(\xi^{\dagger 2}+\xi^2)\\
&+(\xi^\dagger+\xi)(\xi^{\dagger 2}+\xi^2)(\xi^\dagger+\xi)+(\xi^{\dagger 2}+\xi^2)(\xi^\dagger+\xi)^2]\\
&\left.+\frac{(\nu-1)(\nu-2)(\nu-3)}{24\cosh^4\phi e^{-2\phi}}(\xi^\dagger+\xi)^4\right],
\end{aligned}
\tag{19}
$$

with

$$
\begin{aligned}
F=&\frac{1}{2\cosh^2}[\xi^\dagger\xi\xi+\xi^\dagger\xi^\dagger\xi]-\frac{\sinh^2\phi}{8\cosh^2\phi}[(\xi^\dagger+\xi)^2\xi+\xi^\dagger(\xi^\dagger+\xi)^2+2(\xi^\dagger+\xi)\xi^\dagger\xi\\
&+2\xi^\dagger\xi(\xi^\dagger+\xi)]\\
&+\frac{\sinh^4\phi}{8\cosh^6\phi}(\xi^\dagger+\xi)^3+\frac{s}{2\cosh^4\phi}(\xi^\dagger+\xi)
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
G=&-\frac{\sinh\phi}{8\cosh^4\phi}[(\xi^\dagger+\xi)\xi^\dagger\xi\xi+\xi^\dagger\xi(\xi^\dagger+\xi)\xi+\xi^\dagger(\xi^\dagger+\xi)\xi^\dagger\xi\\
&+\xi^\dagger\xi^\dagger\xi(\xi^\dagger+\xi)+2\xi^\dagger\xi\xi^\dagger\xi]\\
&+\frac{\sinh^3\phi}{16\cosh^3\phi}[(\xi^\dagger+\xi)^3\xi+\xi^\dagger(\xi^\dagger+\xi)^3]\\
&-\frac{5\sinh^5\phi}{64\cosh^8\phi}(\xi^\dagger+\xi)^4-\frac{s\,\sinh\phi}{\cosh^4\phi}[(\xi^\dagger+\xi)\xi+\xi^\dagger(\xi^\dagger+\xi)]\\
&-s\,\frac{\sinh\phi}{2\cosh\phi}\xi^\dagger\xi-\frac{s^2\sinh\phi}{\cosh^4\phi}+\frac{3s\,\sinh^2\phi}{8\cosh^6\phi}(\xi^\dagger+\xi)^2
\end{aligned}
\tag{21}
$$

where $h^{(0)}$ is diagonal in the harmonic oscillator basis (i.e., in the basis of $(\xi^\dagger\xi)$) and $h^{(1)}$ and $h^{(2)}$ contain various powers of $\xi$ and $\xi^\dagger$. Therefore it is useful to consider $h^{(0)}$ as the unperturbed hamiltonian and treat $\alpha h^{(1)}+\alpha^2h^{(2)}$ as the perturbation. One may notice that in $h^{(1)}$, $\xi$ and $\xi^\dagger$ appear in odd powers and so the lowest-order contribution from $\alpha h^{(1)}$ should come through the second-order perturbation theory. Thus to order $\alpha^2$, the energy in each angular momentum sector reads

$$E_{nl} = \alpha^{-2}{\epsilon_n}^{(-1)} + {\epsilon_n}^{(0)} + \alpha^2{\epsilon_n}^{(1)} \tag{22}$$

where

$${\epsilon_n}^{(-1)} = \frac{(1+\nu-\cosh^4\phi)}{\nu} \tag{23}$$

$${\epsilon_n}^{(0)} = [(2n+1)\cosh^2\phi + s - 1] \tag{24}$$

$${\epsilon_n}^{(1)} = \frac{e^{-2\phi}}{\omega}\sum_m{}' \frac{|\langle n|h^{(1)}|m\rangle|^2}{({E_n}^{(0)} - {E_m}^{(0)})} + \frac{e^{-2\phi}}{\omega}\langle n|h^{(2)}|n\rangle \tag{25}$$

$|n\rangle$ being an eigenstate of $h^{(0)}$ belonging to the eigen value

$${E_n}^{(0)} = \omega e^{2\phi}[\alpha^{-2}{\epsilon_n}^{(-1)} + {\epsilon_n}^{(0)}] \tag{26}$$

We are interested in the ground state energy calculation for which we put $n = 0$. We then obtain the ground state energy to order $\alpha^2$ in each angular momentum sector:

$$\begin{aligned} E_{0,l} = \mu\Bigg[&\frac{1}{\alpha^2}\left(\frac{1+\nu-\Omega^2}{\nu}\right) + \alpha^0(\Omega+s-1) \\ &+\alpha^2\Bigg(\frac{(\Omega^2-1)(9(\Omega-1)(\Omega-11)-6(\nu-2)(\Omega^2+6\Omega-11)-11(\nu-2)^2(\Omega^2-1)+9(\nu-2)(\nu-3)\Omega^2)}{72\Omega^4} \\ &+\frac{(\Omega^2-1)s(s\Omega+1-2\Omega+\nu)}{2\Omega^3}\Bigg) + \cdots\Bigg], \end{aligned} \tag{27}$$

where $\Omega = \cosh^2\phi$ and $\mu = \omega e^{2\phi}$. The shift parameter $s$ has so far been arbitrary. For $s = 0$ we get back the unshifted result of [56]. Several prescriptions [50–54] are now available for the choice of $s$. We shall follow the novel scheme of Imbo et al. [34] in which one sets the coefficient of the $\alpha^0$-term equal to zero i.e.,

$${\epsilon_n}^{(0)} = 0, \tag{28}$$

which yields (for any arbitrary state in general)

$$s = [1 - (2n+1)\Omega]. \tag{29}$$

For the Coulomb potential, $\mu = e^4$, $\omega_0 = 0$, $\nu = -\frac{1}{2}$ and $s = \frac{1}{2}$. Substituting these values in (27), we find that the first term itself gives the exact result, all the higher-order terms being identically zero. For the harmonic oscillator case also, the first term gives the exact result.

We shall verify our result for the general power law potential $V(r^2) = 2\nu^2 r^{2\nu}$ with $(\omega_0 = 0)$. In this case, we have

$$\omega e^{2\phi} = (4\nu^2 g^2)^{1/(1+\nu)} \tag{30}$$

$$\Omega = \cosh^2\phi = \frac{(1+\nu)}{2}^{1/2}, \tag{31}$$

$$s = 1 - \frac{(1+\nu)}{2}^{1/2}, \tag{32}$$

and the ground state energy is given by

$$E_{0,l} = (N + 2l - 2 + \sqrt{2(\nu + 1)})^{\frac{\nu-1}{\nu+1}} (4e\nu)^{\frac{2}{\nu+1}} \times \left[ \frac{(1+\nu)(N + 2l - 2 + \sqrt{2(\nu+1)})}{8\nu} - \frac{2(2\nu+1)(\nu-1)}{144(N + 2l - 2 + \sqrt{2(\nu+1)})} + \dots \right] \tag{33}$$

which is identical (to order considered in this paper) to the shifted $1/N$ expansion result of Sukhatme and Imbo [33] obtained with the help of the perturbed oscillator method. One can see that for $\nu = 1$ (harmonic oscillator) and $\nu = -\frac{1}{2}$ (Coulomb potential) only the first term survives and one gets the exact results.

To conclude, we have extended in this paper the $SO(2, 1)$ formulation of large-$N$ expansion originally proposed by MP to develop a shifted $1/N$ expansion scheme within the framework of the pseudo-spin formalism. We have shown that this formulation gives exactly the same results for the harmonic oscillator, the hydrogen atom and the general power-law potential problems as obtained from perturbed oscillator method. However the present method has one important advantage over the perturbed oscillator method. As alluded to in the introduction, unlike the latter, the present technique has a much wider applicability. For example, the $SO(2, 1)$ formulation can be applied to several fields of physics. The potential areas are field theory, statistical mechanics and condensed matter physics. It has already been shown that the large-$N$ expansion developed within the framework of path- integral formalism can be considered as a saddle-point expansion in $1/N$. MP have shown that the leading-order term of this expansion in the $SO(2, 1)$ pseudo-spin formulation yields the mean-field BCS gap equation of superconductivity. Thus one would expect that the shifted-large-$N$ expansion developed within the framework of the $SO(2, 1)$ formalism, can possibly capture some of the quantum fluctuations even in the leading-order of the saddle-point expansion. This is a big and palpable advantage since calculating higher-order terms in the saddle-point expansion is normally a difficult job indeed.

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